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非弾性衝突の数値シミュレーション
The Simulation of the Inelastic Impact

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1 Introduction

Collisions are common phenomena in nature. For example, in the microscopic scale, atoms and molecules in gas are colliding each other. In the macroscopic scale, we often see collision of balls in sports such as the baseball and the billiard. In such collisions, the initial kinetic energy of material dissipates into internal degrees of freedom like elastic vibration, sound emission, and heat. As a result, macroscopic collisions are always inelastic.

Inelastic collisions play an important role in granular materials[1]. Characteristic behaviors of granular material come from inelastic collisions among particles. By tilting or shaking the container which contains granular material, one can see the characteristic behavior of granules which is different from that of ordinary fluid. The *Distinct Element Method* (DEM) is a well-known simulation method for the granular materials[2]. DEM contains some phenomenological parameters such as the Coulomb's coefficient of friction, dashpots, and so on. Nobody can determine such the viscoelastic parameter from the first principle. However, even the determination of the simplest parameter, the coefficient of normal restitution (COR) is not reliable.

The coefficient of normal restitution(COR) e is a familiar parameter which is introduced in text books of the elementary physics. COR is defined by the ratio of the normal components of the initial collision velocity v_i and the rebound velocity v_r as

$$e = -v_r/v_i, \quad 0 \leq e \leq 1. \tag{1}$$

Historically, COR was first introduced by Newton[3]. Though many text books of elementary physics state that COR is a material constant, many experiments and simulations show COR decreases as the impact velocity increases[4, 5, 6, 7, 8, 9, 10, 11]. On the other hand, Louge and Adams reported in their recent paper that COR e can exceeds unity in the situation of the oblique impact which is contrary to the assumption $e \leq 1$[12]. This topic is interesting and worthy of more detailed study.

In addition, the coefficient of tangential restitution β is also well-known parameter to describe the rotational motion of material. β is defined as

$$\beta = -\frac{v_t'}{v_t}, \tag{2}$$

where v_t and v_t' are the tangential components of the velocity of the contact point before and after collision. β is known to be dependent on the incident angle of impact. However, the mechanism of this dependency is not unclear.

Our research is to understand the mechanism of the coefficient of tangential restitution. We study the relation between the coefficient of tangential restitution and the angle of incidence in oblique collision in this paper. The organization of this paper is as follows. In the next section, we will review the definition of the coefficient of restitution and the coefficient of tangential restitution. In section 3, we introduce our numerical model and setup of the simulation. Section 4 is the main part of this paper where we summarize the results of our simulation and, explain the numerical results by the theory. Section 5 is the conclusion of this paper.

2 Introduction of e and β

To characterize inelastic collision, Walton introduced three parameters[13]. The three parameters are the coefficient of normal restitution e, the coefficient of Coulomb's friction μ, and the maximum value of the coefficient of tangential restitution β_0. Experiments have supported that his characterization adequately capture the essence of binary collision of spheres or collision of a sphere on a flat plate[14, 15, 16, 17]. Now, let us define the coefficient of restitution e and the coefficient of tangential restitution β in the 2-dimensional situation. Figure 1 is the schematic figure that a disk

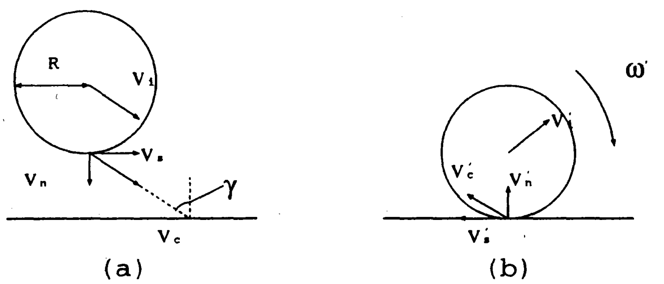


Figure 1: The schematic figure of a collision of sphere with a wall.

is colliding with a stationary wall with initial velocity of its center of mass, $\mathbf{v_i}$. The relative velocity at the contact point after collision, thus, becomes

$$\mathbf{v}_c' = \mathbf{v}_i' - R\mathbf{n} \times \boldsymbol{\omega}', \tag{3}$$

where R is the radius of the disk, $\mathbf{n}$ is the unit vector in the normal direction to the wall, and $\boldsymbol{\omega}'$ is the angular velocity. The prime denotes post-colliding quantities. The coefficient of normal restitution e is defined as

$$\mathbf{v}_c' \cdot \mathbf{n} = -e\mathbf{v}_c \cdot \mathbf{n}. \tag{4}$$

Conventionally, this parameter is assumed to be $0 \leq e \leq 1$.

The coefficient of tangential restitution β is defined as

$$\mathbf{v}_c' \cdot \mathbf{t} = -\beta\mathbf{v}_c \cdot \mathbf{t}, \tag{5}$$

where $\mathbf{v}_c'$ and $\mathbf{t}$ are the post-collisional velocity at the contact point after collision and the unit tangential vector, respectively. It is believed that β is a function of the angle of incidence γ, with possible values lying in the range between -1 and 1 [13, 14]. The incident angle γ is defined as $\gamma = \arctan(v_t/v_n)$, where v_n and v_t are $v_n = \mathbf{v}_c \cdot \mathbf{n}$ and $v_t = \mathbf{v}_c \cdot \mathbf{t}$, respectively.

For the oblique collision, the coefficient of tangential restitution β is more important than e. From the conservation laws of momentum and angular momentum and Coulomb's friction on the surfaces of two identical rigid spheres, Walton[13] derives

$$\beta \simeq \begin{cases} -1 - \mu(1+e)\cot\gamma\left(1 + \frac{mR^2}{I}\right) & (\gamma \geq \gamma_0) \\ \beta_0 & (\gamma \leq \gamma_0), \end{cases} \tag{6}$$

where γ_0 is the critical angle, and m, R, and I are mass, radius and moment of inertia of spheres respectively. Labous, Rosato, and Dave performed the experiment of binary collision of nylon spheres and showed the consistency of their results to the Walton's relation[14]. Furthermore, it has become clear that Many experimental results are consistent with the relation so that Walton's model is accepted as reasonable[15, 16, 17]. Meanwhile, Maw, Barber, and Fawcett extended the Hertz theory of impact and established the theory of the oblique impact to be consistent with their experimental results[18]. In contrast to Walton's assumption, they demonstrated the need to consider normal and tangential compliance over the contact area.

3 Our Models

Here, let us introduce three lattice models. Each model consists of an elastic disk and an elastic wall. The main results of this paper are those of random lattice model(Fig. 2). Both the disk and

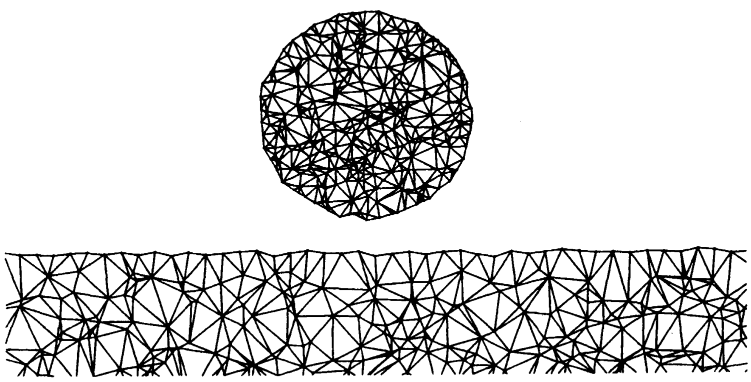


Figure 2: The elastic disk and wall consisted of random lattice system.

the wall are composed of randomly distributed 800 mass points. All mass points are bound with nonlinear springs using the Delaunay triangulation algorithm[19]. The spring interaction between connected mass points is described as

$$V(x) = \frac{1}{2}k_a x^2 + \frac{1}{4}k_b x^4, \tag{7}$$

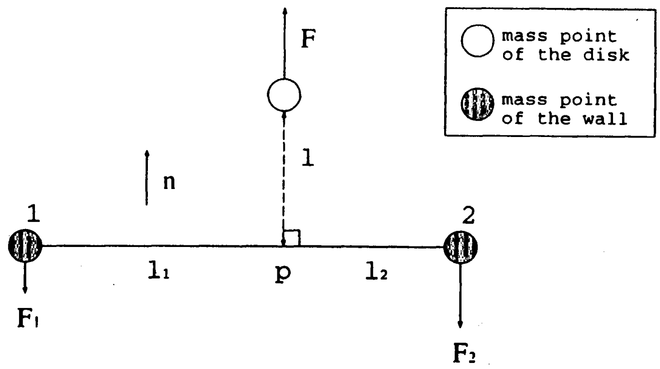


Figure 3: Interaction between surface particles of the disk and the wall.

where x is a stretch from the natural length of spring, and k_a and k_b are the spring constants. We use a typical ratio of k_b to k_a as $k_b/k_a = 10^{-3}$. The width of the wall is 4 times as long as the diameter of the disk. The height of the wall is same as the diameter of the disk. Two sides of the wall are fixed.

The interaction between the disk and the wall during a collision is introduced as follows. Figure 3 is the schematic figure of the interaction of surface mass points of the disk and the wall. When the distance l between the edge of the disk and the surface of the wall is less than the cutoff length(we set it equal to the length of the linear spring), the surface particles of the disk feel the repulsive force, $\mathbf{F}(l) = aV_0 \exp(-al)\mathbf{n}$, where a is $300/R$, V_0 is $amc^2R/2$, m is the mass of the particle, R is the radius of the disk, $c = \sqrt{E/\rho}$, E is Young's modulus, and ρ is the density, $\mathbf{n}$ is the normal unit vector to the surface. The reaction forces applied to the two points of the surface of the wall (point 1 and 2) are decided by the balance of the torques as $\mathbf{F}_1(l) = -F(l)\mathbf{n}/(1 + l_1/l_2)$ and $\mathbf{F}_2(l) = -F(l)\mathbf{n}/(1 + l_2/l_1)$, where $l_i (i = 1, 2)$ is the distance between the point p and the point i (see Fig. 3).

In this model, roughness of the surfaces is important mechanism to make the disk rotate after collision. How to make roughness is as follows. At first, we generate normal random numbers whose average value is 0 and then make the initial position of particles on surface of both the disk and the wall deviate with them. We choose the value of dispersion δ as $\delta = 3 \times 10^{-2}R$, where R is the radius of the disk.

As for random lattice model, we cannot determine Poisson's ratio theoretically. When we determine Poisson's ratio of this model, we introduce the viscous damping term in (7). By stretching the strip of random lattice and measuring its width and height, we can obtain Poisson's ratio.

For comparison, we make other two lattice models: triangular lattice and square lattice disk(Fig.4). The triangular lattice disk is made by replacing the internal structure of the random lattice disk with the triangular lattice. The surface of the triangular lattice disk is same as that of the random lattice disk. Poisson's ratio of the triangular lattice can be calculated theoretically as 1/3[20]. The square lattice disk is made by replacing the internal structure of the random lattice disk with the square lattice. We introduce two spring constants: $k_a = k_1$ for nearest neighbor interaction and $k_a = k_2$ for next-nearest neighbor interaction. Poisson's ratio of the square lattice is expressed as

$$\nu = \frac{k_2^2 + (k_1^2 - 4k_2^2)n_x^2n_y^2}{k_2(k_1 + k_2) + (k_1^2 - 4k_2^2)n_x^2n_y^2}. \tag{8}$$

We scale the equation of motion for each particle using the radius of the disk R as the scale of

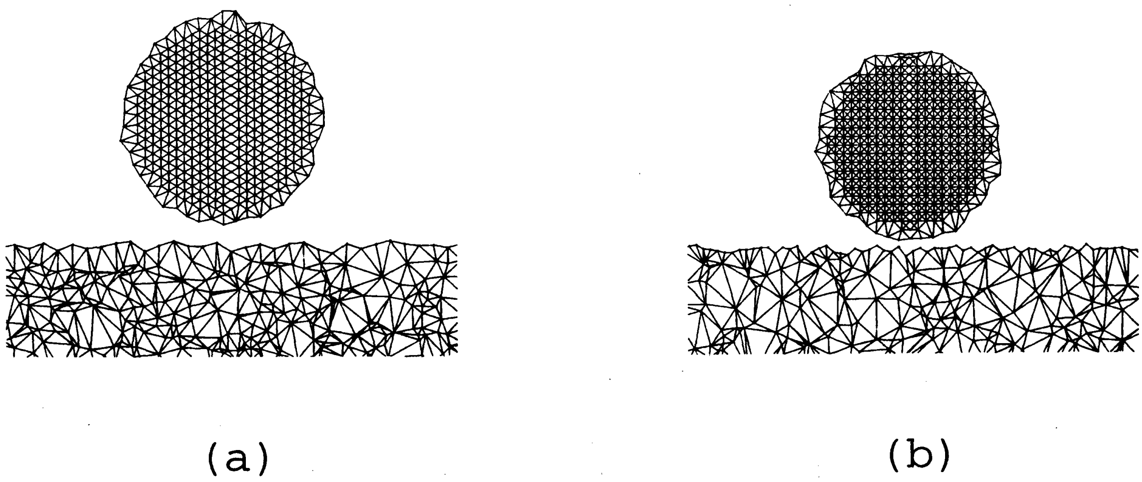
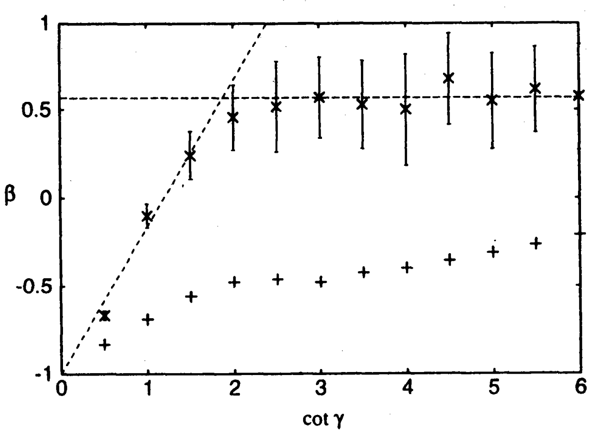


Figure 4: The schematic figures of (a) triangular lattice disk and (b) square lattice disk.

length and the velocity of elastic wave $c = \sqrt{E/\rho}$ as the scaling unit of velocity. As the numerical scheme of the integration, we use the fourth order symplectic numerical method with the time step $\Delta t \simeq 10^{-3}R/c$.

4 Results and Discussions

In this section, we carry out the simulation of the oblique impact. The angle of incidence is ranged from 5.7° to 80.5° while the normal component of velocity is fixed as $0.1c$. The disk has no internal vibration and rotation before collision. In order to eliminate the effect of the initial configuration of mass points, we prepare 100 samples of disk as the initial condition by using 100 sets of random numbers and average data of all samples.



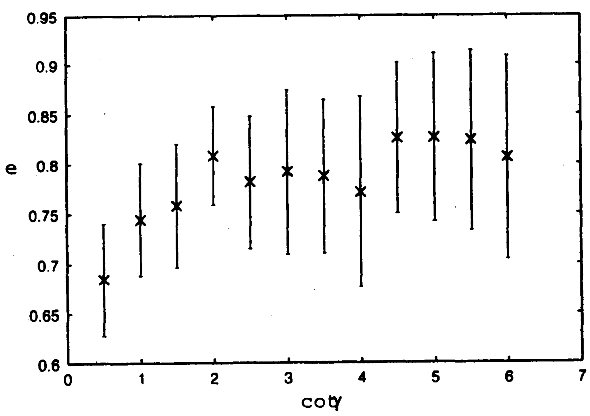


Figure 5: The relation between $\cot\gamma$ and β. Figure 6: The relation between $\cot\gamma$ and e.

Figure 5 shows the relation between the cotangent of the angle of incidence γ and the coefficient of tangential restitution β. In this figure, cross points are the result of the random lattice disk and wall, and broken lines are eq.(6), where $e = 0.8$, μ and β_0 are fitting parameters. This result shows that β_0 takes the value nearly 0.56 and μ_0 takes the value nearly 0.18. From this estimation, we can see that this model can reproduce the tendency of the experimental results of the oblique collision qualitatively[14, 15]. In contrast, plus points are the results of the triangular lattice disk. In this model, β takes negative values in all range of the angle of incidence. This means that the triangular lattice model is easy to slip on the surface.

Figure 6 shows the relation between the cotangent of the angle of incidence and COR e. Although it is expected that COR takes the constant value because the normal velocity of the disk is set to the fixed value, $0.1c$, COR depends on the angle of incidence. In particular, in the region of small value of $\cot\gamma$, COR decreases as $\cot\gamma$ decreases. At present, we cannot explain this tendency of normal COR.

Here, we compare our result with the theory of Maw, *et al.*[18]. According to their theory, all the region of the angle of incidence can be divided into three regimes. For each regime, β can be expressed as

(i) $1/\mu\eta^2 < \cot\gamma$:

$$\beta = \cos\omega t_1(\gamma) + \mu\alpha e \left[1 + \cos\left(\frac{\Omega t_1(\gamma)}{e} + \frac{\pi}{2}(1 - e^{-1})\right)\right] \cot\gamma, \tag{9}$$

(ii) $\mu(1+e)/\alpha < \cot\gamma < 1/\mu\eta^2$:

$$\beta = \cos\omega t_3(\gamma) + \mu\alpha \left[1 + e - \frac{p(t_3(\gamma))}{p(t_c)}\right] \cot\gamma, \tag{10}$$

(iii) $\cot\gamma < \mu(1+e)/\alpha$:

$$\beta = 1 - \mu\alpha\,(1+e)\cot\gamma, \tag{11}$$

where μ is the coefficient of friction, η is the constant dependent on Poisson's ratio, $\alpha = 3.02$ which is a constant dependent on the shape of material, $\Omega = \pi/2t_c$, t_c is a duration of a collision, $\omega = (\pi/2\eta t_c)\sqrt{\alpha}$, $t_1(\gamma)$ is the transition time from stick motion to slip motion, $t_3(\gamma)$ is the transition time from slip motion to stick motion, and $p(t)$ is impulse. This theory was confirmed to be consistent with experimental data[15, 16, 17, 18].

We compare the result of simulation of the oblique impact using the random lattice model with the theoretical curve(Fig. 7). Here we used $\eta = 1.015$, which corresponds to Poisson's ratio 0.058, $e = 0.8$ as a fixed value, and $\mu = 0.3$ as a fitting parameter. It is found that the result of random lattice model is consistent with the theory.

On the other hand, as for the result of figure 5, we focus our attention to the difference of Poisson's ratio between the random disk and the triangular disk. By changing the value of spring constants of square lattice disk and controlling Poisson's ratio, we investigate the dependency of β_0 on Poisson's ratio. Figure 8 is the result when $\nu = 0.1$ while figure 9 is the result when $\nu = 0.3$. We cannot see the difference of the values of β_0. From these results, Poisson's ratio seems not to affect the value of β_0.

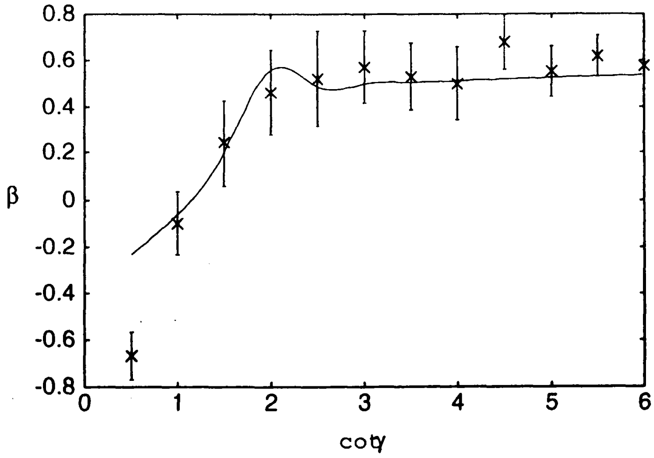
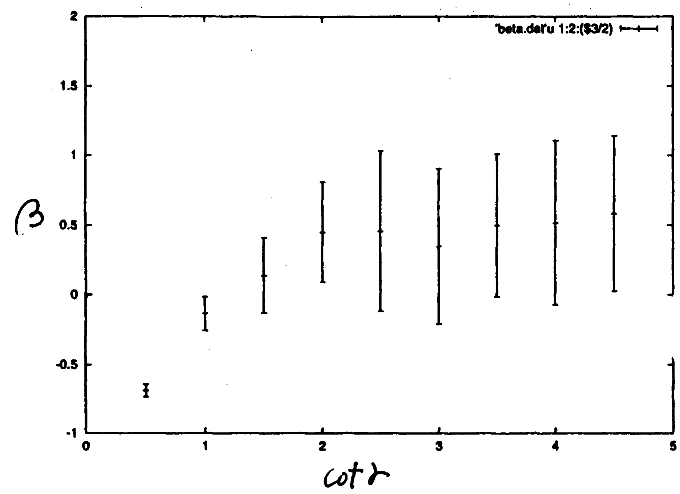


Figure 7: The relation between $\cot\gamma$ and β. Cross points are the numerical results of the random lattice model. Solid line is the theoretical curve.



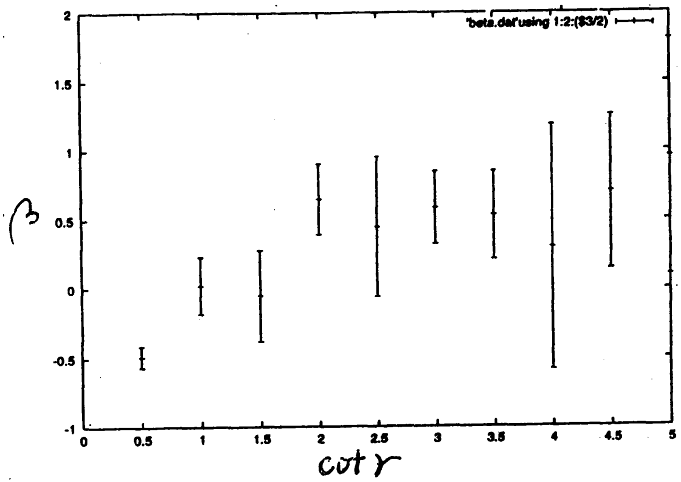


Figure 8: The relation between $\cot\gamma$ and β. when $\nu = 0.1$

Figure 9: The relation between $\cot\gamma$ and β. when $\nu = 0.3$

5 Conclusion

In this paper, we demonstrate the 2-dimensional simulation of the oblique impact and obtain results as follows.

(i) Our random lattice model exhibits the same tendency as experimental data qualitatively. In addition, the model is consistent with Maw's theory of the oblique impact.

(ii) There seems to be no relation between Poisson's ratio of material and the value of β_0.

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